THE AMERICAN MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. XIX. AUGUST-SEPTEMBER, 1912. NOS. 8-9.

A LABOR-SAVING DEVICE FOR SERIAL MULTIPLICATION OR DIVISION BY MEANS OF AN ARITHMOMETER IN CASES WITH SMALL DIFFERENCES OF CONSECUTIVE RESULTS.*

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INTRODUCTION.

In computing the present value of future expense charges on annual dividend policies for a life insurance company in New York, I had to multiply a constant for every five years of age, representing a certain percentage of the mean of the office premium and the net premium for ordinary life, by a decreasing annuity for the different policy ages. The results obtained were taken only to the nearest integer. As there happened to be repetitions of some integers and omissions of others with what seemed to be absolute irregularity, I thought of trying to determine the limits between the factors giving one integer and those giving the next. This led me to devise a method of serial multiplication by means of the arithmometer (later extended also to serial division) for such and similar kinds of work, which eliminates most of the labor required in the ordinary methods if the differences of the consecutive results do not exceed a few units,—and a large proportion of it for differences greater than a few units. The present paper is an explanation of this device, which, I believe, may prove of interest also to other actuarial computers, as well as to mathematicians in general.

SERIAL MULTIPLICATION.

I shall first treat of serial multiplication when used for finding the nearest or the highest integers of a series of consecutive products with a constant factor, whose differences do not exceed a few units — the actual case that gave rise to the discovery of the device.

*Presented to the American Mathematical Society at its April meeting.

Let $x_{0,1},\ x_{0,2},\ \ldots,\ x_{0,\,k},\ \ldots;\ x_{1,1},\ x_{1,2},\ \ldots,\ x_{1,\,k},\ \ldots;\ \ldots;$
$x_{i,1},\ x_{i,2},\ \ldots,\ x_{i,k},\ \ldots;\ \ldots$

represent the required products, in which the first subscripts indicate their nearest or their highest integers, as the case may be; and let

$$a_{0,1},\ a_{0,2},\ \ldots,\ a_{0,\,k},\ \ldots;\ \ldots;\ a_{i,1},\ a_{i,2},\ \ldots,\ a_{i,k},\ \ldots;\ \ldots$$

be the corresponding given values of the variable factor, while m is the constant factor. So that we have,

$$x_{0,1}=ma_{0,1},\ \ldots,\ x_{0,\,k}=ma_{0,\,k},\ \ldots;\ \ldots;\ x_{i,1}=ma_{i,1},\ \ldots,\ \ldots$$

and, in general, $x_{i.k}=ma_{i,k},\ \ldots$

Hence

$$a_{i,k}=\frac{1}{m}x_{i,k} \qquad (1).$$

Also,

$$x_{0,\,\mathrm{k}}<.5 \text{ (or 1)} \leqq x_{1,\,k}<1.5 \text{ (or 2)} \leqq x_{2,\,k}<2.5 \text{ (or 3)} \qquad (2)$$

— the double symbol $\leqq$ having reference to the value of $k^*=1$ only, and reducing to the single symbol $<$ for all other values of k.

Multiplying (2) by $1/m$ and substituting from (1), we obtain

$$a_{0,k}=\frac{1}{m}x_{0,k}<.5\times\frac{1}{m} \text{ (or } 1\times\frac{1}{m}) \leqq \frac{1}{m}x_{1,k}=a_{1,k}<1.5\times\frac{1}{m} \text{ (or } 2\times\frac{1}{m})$$
$$\leqq \frac{1}{m}x_{2,k}=a_{2,k} \ldots \text{ etc., } \ldots (3),$$

the double and the single symbols having the same reference as before; or simply,

$$a_{0,k}<.5\times\frac{1}{m} \text{ (or } 1\times\frac{1}{m}) \leqq a_{1,k}<1.5\times\frac{1}{m} \text{ (or } 2\times\frac{1}{m}) \leqq a_{2,k},\ \ldots,\ \text{etc.};$$

and, in general,

$$a_{i,k}<\frac{1}{m}(i+.5) \leqq a_{i+1,k}<\frac{1}{m}(i+1+.5) \qquad (4a);$$

*In this notation, k merely marks a certain number of the set with which we are dealing, and it is convenient to use $k=1$ to mark the first number considered, whose nearest or highest integer is i.

for the case of the nearest integers, or

$$a_{i,k} < \frac{1}{m}(i+1) \leqq a_{i+1,k} < \frac{1}{m}(i+2) \tag{4b}$$

for the case of the highest integers,—for all values of i from 0 up (where the the sign of equality of the double symbol *may* hold only for $k=1$).

Hence, $\frac{1}{m}(i+.5)$, or $\frac{1}{m}(i+1)$, are the critical values for locating all the lower values of $a_{i,k}$, which produce an integer i *at most*, and all the higher values of $a_{i,k}$, which produce an integer $i+1$ *or greater*. Moreover, all the values of $a_{i+1,k}$, giving the integer $i+1$ (for $i=0$, 1, 2, ..., etc.), lie between $\frac{1}{m}(i+.5)$ and $\frac{1}{m}(i+1.5)$ for the nearest integer case, or between $\frac{1}{m}(i+1)$ and $\frac{1}{m}(i+2)$ for the highest integer case,—with the very rare possibility of *one* of these values equaling the lower limit for $k=1$.

What we have to do, therefore, to find the a's (if any) producing a given integer $i+1$, is to multiply a sufficient approximation of $1/m$ successively by $i+.5$ and $i+1.5$, in the one case, and by $i+1$ and $i+2$, in the other, and assign the product $i+1$ to all the a's between the obtained products. There may be several such a's, in which case we save the labor of multiplying each of them by m in the direct process; and there may be no a whose value lies between the obtained limits, thus showing that there could be obtained no integer $i+1$ in the direct process of multiplying the a's by m. In the latter case we add the value of $1/m$ once, twice, etc., times to the higher of the limits last obtained, until we get the first of the products, $\frac{1}{m}(i+1.5+h)$ or $\frac{1}{m}(i+2+h)$, just exceeding some $a_{i+1+h,1}$, or (very rarely) just equal to some $a_{i+2+h,1}$. Then all the a's immediately below this product and above the limit previously obtained, *i. e.*, $a_{i+1+h,1}$, $a_{i+1+h,2}$, ..., $a_{i+1+h,l}$, give the integer $i+1+h$, while $a_{i+2+h,1}$ gives the integer $i+2+h$ in the very rare case of exact equality. In either of the last mentioned two possibilities, $i+1+h$ or $i+2+h$, is the first integer above i that could be obtained also in the direct process of multiplication.

EXAMPLES.

The process is best illustrated by one or two concrete examples:

Let $m=5.135$ and let the variable factors in their order of magnitude be those given in the following table, which for convenience is broken up into six partial columns:

1	2	3	4	5	6
.131	.290	.506	.902	1.617	2.789
.158	.326	.569	1.017	1.815	3.078
.189	.365	.637	1.143	2.031	3.383
.221	.403	.710	1.281	2.266	3.705
.255	.451	.800	1.439	2.519	

Let us further suppose that the computer is seeking the nearest integers of the corresponding products and is using a "Saxonia" arithmometer for the purpose. The actual products of the whole of his work lying within a range less than 100, it is sufficient to take $1/m=.19574$, exact to the fifth decimal place, since $a_{i,k}=\frac{1}{m}\times x_{i,k}<\frac{1}{m}\times 100$, which will make $a_{i,k}$ correct to the nearest third decimal place, as given in the table of values of a's. Now, putting.19474 on the fixed plate of the "Saxonia," and multiplying it by .5, we obtain .097370, and taking only the needed three decimal places .097, we find it less than the smallest $a=.131$. Therefore, we get no values zero for any of the nearest integers sought. Next we move over the slide one station to the right, efface the 5 in the first "quotient hole," but leave the original product on the slide, and turning the handle once, we get .292110, or .292>.290 and all the five given a's below it. Therefore, we write 1, the number in the second quotient hole from the right, opposite every one of the six lowest a's. We throw up the multiplicand once more on the slide, and we get .48685, or .487>.451>.403>.365>.326. Hence we write 2 opposite these four numbers. Similarly, we get by another turn of the motive handle, 3 opposite the next higher three numbers, and by still another turn 4 opposite the next two and so on until for nine turns we obtain the nearest product 1.850>1.815, giving nine for this factor. We turn the handle again, and we get 8, colored red,* in the quotient hole. This gives the computer warning that the 8 in the second quotient hole really corresponds to the tenth turn of the handle additively. The nearest number obtained in the product is 2.045, giving 10 opposite the given $a_{10,1}=2.031$. The operator could go on turning the handle and obtaining red 7, 6, 5, etc., in the reversed order, and consider them equivalent to black 11, 12, 13, etc.— for which we have the rule that each of the first red figures, f_r, turned up additively immediately after the first nine black figures, is equivalent to $f_r+2(9-f_r)$ given in black; thus, $5_r=5+2(9-5)=13$, etc. But it is better, as soon as the operator receives the warning of the first red figure, to efface it in its quotient hole, and put down the equivalent number in black color; after which he may go on turning the handle addi-

* In the "Saxonia" arithmometer the figures showing how many times a number has been taken subtractively are distinguished by the red color from those showing how many times a number is taken additively, which are given in black.

tively, obtaining black numbers again until we reach 19, followed again, in the next turn, by 1 black in the tens and 8 red in the units, which is equivalent to twenty turns. Here once more the rule would hold that a number with red units is equivalent to the same number plus twice the excess of 19 over it; or in symbols, $n_m = n_b + 2(19 - n_b)$, where n_m is a number in mixed color immediately after the first nineteen additive turns, and n_b is the same number in purely black color, *e. g.*, $17_m = 17_b + 2(19 - 17_b) = 21_b$.

In our case the computer will find that the nearest product corresponding to ten turns is 2.045, giving 10 as the nearest integer derived from the factor 2.031. The nearest product corresponding to eleven turns is $2.240 < 2.266$. Hence, between $\frac{1}{m} \times 10.5 = 2.045$ and $\frac{1}{m} \times 11.5 = 2.240$ there is no factor, the two consecutive given factors being 2.031 and 2.266, one lower than the lower limit, and the other higher than the higher limit. We therefore turn the handle once more, finding the next nearest product $2.434 > 2.266$, giving 12 for this factor. If our table should also contain the factors 2.320, 2.390, and exactly 2.434, the integer 12 would correspond also to the first two additional factors, but the last factor would give 13 as the nearest integer in the direct process, and, as has been proven above, must also be assigned 13 as its nearest product in our work.

On the other hand, if our table of given factors would miss the factors 1.617, 1.815, 2.031, 2.266, the factors, namely, corresponding to the integers 8, 9, 10, 12, then, after having thrown up on the slide 1.46055, giving the integer 7 for the factor 1.439, we should have to turn the handle continuously six times more to get the nearest product 2.629, the first to exceed the given factor 2.519, which would, therefore, have 13 for its nearest product. Now, in such a case, while turning the handle and keeping our eye open upon the products thrown up until we would notice one greater than 2.519, the factor next higher than 1.439, we would naturally be likely to overlook the last 9 in black color in the "quotient holes" or even perhaps the first red figures 8, 7, 6, until we came to red 5. The rule, therefore, that $5_r = 5_b + 2(9 - 5_b) = 13_b$ is of advantage for such a case, as it enables us to relieve our attention for a while from the "quotient holes," and fix it wholly upon comparing the thrown up consecutive products with the column of given factors.

We see from our example that, besides the time spent in obtaining $1/m$ to five decimal places and in setting upon the machine the quotient .19474 and making the five preliminary turns to insure correctness in the third decimal place, we have only to make nineteen turns of the handle and, in addition, to replace a red figure by its equivalent black number, in order to enable us to write down all the required integers corresponding to the given twenty-nine factors, or to a similar series of factors of which the last one is less than 3.79743. Whereas in the direct process of multiplication

we would need, after setting up 5.135, the constant factor, to multiply it by the twenty-nine given co-factors, each containing from three to four figures, or an average of 101 figures (by actual count in this example 100 figures), each figure requiring, on an average, $\frac{0+9}{2}=4\frac{1}{2}$ turns, which makes in the aggregate about 455* turns of the motive handlé, besides the twenty-nine effacings of the products, in order to find the same twentynine integers. Of course, some time might be gained also in the direct process of multiplication, by forming the differences of the factors; but not very much, since these differences are not always small, growing up in our example to .322 (=.705—3.383), besides involving the risk of carrying any error in the middle of the process to the very end. We see, therefore, that, discounting the time spent in registering the integers, which is in the two processes the same, our indirect process, which may be called *the index process* (as the proper first indexes of the different $a_{i,k}$'s are directly thrown up on the slide in the "quotient holes") would be *at least* about twenty-four times shorter than the direct process, namely, $\frac{455}{19}=24$, even neglecting the twenty-nine effacings, and about ten or twelve times shorter than by the process of differences.† Moreover, after a little practice with the index method one is enabled with great ease to turn the handle with the left hand and do the registering of the obtained integers beside the given factors with the right hand, in the case where each factor has another integer belonging to it, and where some integers may be entirely missing, where, consequently, we have to turn the handle at least once and, frequently, even twice, three, or four times, before we arrive at the integer belonging to a new factor. The kind of work where this occurs is just as frequent in practice as the kind of work represented by our example, where one integer belongs to several factors.

In all, I think, it is safe to assume that the index process would save at least seventy-five to ninety-five per cent of the work (excluding the registering); and more frequently the higher percentage of the work is saved. Besides, practice will show that it is also a very safe and reliable process, much less subject to error than either the direct or the difference process.

In our example we have supposed that we were looking for the nearest integers. In case the highest integers are sought, the process will be in all respects identical, except that, instead of multiplying originally $1/m$ by .5, effacing the 5 in the first "quotient hole" of the slide, and moving the slide over one station to the right, in which the indexes 1, 2, etc., are turned up, we have to start with multiplying $1/m$ by 1, and, effacing it, proceed directly to turn up the required indexes in the first "quotient hole."

* In the actual example only 373 turns are needed, as the co-factors happen to abound in 0's, 1's and 2's. Other similar series of co-factors might happen to abound in 7's, 8's, and 9's, increasing the number of turns in the direct process, without, however, affecting the differences of consecutive results.

† It is hardly necessary to explain why in the fraction $\frac{455}{19}$, we must neglect the preliminary five turns, as these would also be sufficient even for a series of one hundred factors instead of twenty-nine, besides neglecting all the effacings in the direct process, involving a much greater waste of time.

Serial Division.

Looked at from another point of view, the index process just explained gives us a short method of obtaining the nearest or highest integers of $x_{i,k}=a_{i,k}\div\frac{1}{m}$, or the nearest or highest integers of the quotients obtained in dividing the different a's by $1/m$. It will, therefore, easily be seen that this method can be used with great advantage also for performing a series of divisions in which the divisor is a constant and the consecutive quotients differ from each other by a few units. Then we treat the divisor as we have treated $1/m$ in multiplication, putting it on the board of the arithmometer to the right, and proceeding to multiply it in exactly the same manner as in the case of multiplication, until we get on the slide the first number above the lowest dividend, and then the first number above the next higher dividend, etc. The corresponding multipliers appearing in the quotient holes will be the required quotient integers. Obviously in this case the method might more properly be called the "checking method of division," as the proceeding is virtually the same as would be followed in checking the original quotients by multiplying the constant divisor by each of them and comparing the results with the corresponding dividends.

In trying to extend the field of usefulness of this method by applying it to divisions with larger differences, it will be found that when these latter are too large (exceeding three or four figures), its use as an independent initial method would be of doubtful practical utility, as too many precautions would be requisite to insure accurate results. Yet for *checking* work of this character done by the direct method, it can be used with great speed and expedition, as no special precautions are necessary in this case. and the labor of putting up each new dividend on the sliding plate, as well as of effacing the quotients, would be wholly saved.

Extended Application of the Method.

There is, however, a large field of actuarial work, comprising divisions with but moderately large differences, not exceeding two or three figures, where the application of our "checking method" of serial division used as an independent, initial method, would prove of decided advantage. The actuarial field referred to consists of the kind of computations represented by Mr. George King's conversion and valuation tables appended to his paper "On Policies with Deferred Participation in Profits, and Policies with contingent Bonuses," read before the Faculty of Actuaries in Scotland, and printed in the Transactions of the Faculty, Vol. V, Part IX, No. 53, 1911. The peculiar feature of this work, computed to the third decimal place, is the comparatively small differences of the consecutive results, which may be obtained as the quotients of a number of separate series of divisions, each with the same divisor. Table II, *q. v.*, may serve as an in-

stance, giving the factors for converting immediate cash bonuses into endowments maturing at the end of the deferred bonus term. The individual members of this table are the reciprocals of pure endowments, ${}_tE_x{}^{-1}=D_x\div D_{x+t}$, where x is the age attained and t is the remaining bonus term, D_{x+t} being constant for each series of divisions.

Mr. King recommends the method of reciprocals and differences as the best and speediest for the computation of these tables, and this would, of course, imply the series with D_x constant for each, instead of D_{x+t}. This method is undoubtedly the best of all known up to date. A much greater saving of labor might, however, be effected through the application of our "checking method of serial division," with one or two simple precautions. In the first place the computation of the reciprocals of the divisors, which for different tables are different, including such compound expressions as $M_{x+t}-M_x+D_z$, would be wholly dispensed with, as well as the finding of their differences. In the second place, the differences of the consecutive results, which in our method are the consecutive multipliers, would be found to be considerably smaller than those of the reciprocals of the divisors; thus, while the former mostly range below 100, and rise to a few hundred only at high ages combined with long periods, which is hardly normal, the latter range mostly several hundred, and rise even above several thousand at corresponding high ages and periods, if calculated to insure results correct to the third decimal place; there would, consequently, result a further considerable reduction of labor in the process of arithmometer multiplication.*

An Illustration of the Method.

It will perhaps not be considered superfluous to state in detail the peculiar features of the procedure in this case, where the differences of the results consist of two or three figures:

Placing the divisor, in our example D_{x+t}, on the arithmometer board to the right and multiplying it by .5 of .001 to insure the correctness of the results to three decimal places, we move over the sliding plate four stations to the right, since $D_x\div D_{x+t}$ will give for normal values of x and t an improper fraction less than 10, which, consequently, has, besides

$$*\,(D_x-D_{x+1})\div D_{x+t}=[(D_x-D_{x+1})\div D_x]\times\frac{D_x}{D_{x+t}}=[1-a_{x\,\overline{1|}}]\dagger\times\frac{D_x}{D_{x+t}},$$

taken correct to the third decimal place, is the formula for the differences of consecutive results giving mostly *two significant figures*, and, at most, three for high ages and long periods; and

$$\frac{1}{D_{x+t+1}}-\frac{1}{D_{x+t}}=\frac{D_{x+t}-D_{x+t+1}}{D_{x+t}\times D_{x+t+1}}=\frac{1-a_{x+t\,\overline{1|}}}{D_{x+t+1}}$$

taken correct to the eighth decimal place, is the formula for the differences of consecutive reciprocals, giving three significant figures so long as D_{x+t+1} consists of five figures and $1+a_{x+t\,\overline{1|}}$ remains below 1, and four significant figures when D_{x+t+1} is reduced to four figures only in the integral part, while $1-a_{x+t\,\overline{1|}}$ approaches the value of .1, that is, when $x+t+1$ becomes equal to 70 or thereabout, and around 80 the number of significant figures becomes five. Of course, the O^m Table at three per cent of the British Offices Life Tables, 1893, is used by me for for illustration,—this being the same used by Mr. King in his computation.

† $\overline{1|}$ is used for the symbol $\overline{1}|$. Ed. F.

the three decimal places required, one figure in the integral part, or four figures in all. We then multiply the divisor from left to right by a number just sufficient to make the product exceed the lowest D_x, corresponding to the highest age of the table, which will be the lowest quotient of the series for the given divisor. To obtain this number we turn up in the first to the left quotient hole sufficient units to make the product just exceed the given dividend D_x, and turning then the handle subtractively once, we move over the sliding plate one station to the left, and here again we turn up sufficient units, of the next lower order, to make the product just above the given dividend, turning then again the handle backwards once, and so on, until we obtain the figure in the fourth from the left quotient hole which will just make the product exceed our dividend. This will be the proper figure of the third decimal place. After this we work only the third and second quotient holes counting from right to left, corresponding to the second and third decimal places of the difference between the result previously obtained and the result corresponding to the next higher dividend D_{x-1}. These two figures and each of the subsequent sets of two figures are obtained in the same manner as the original four figures, by working from left to right and correcting each figure of the higher order (of the second decimal place) by turning the handle subtractively once so as to obtain a product just lower than the corresponding dividend. The only precautions requisite to insure correct results are: first, that we work the motive handle only additively, except when correcting any of the figures of the higher orders by turning subtractively once, so as to pass from a figure giving a product just above a given dividend to one giving a product just below it; second, that we correct an occasional red-figured number n_r by a corresponding black-figured number according to the rule, n_r = arithmetical complement of (n_r+2) to the next higher unit + one such unit.*

A Concrete Example.

A concrete example will show that the method in practice works out even easier than in theory:

Starting with $D_{46} \div D_{46}$, for our Table II, *i. e.*, with the divisor $D_{x+t} = D_{46}$ and $t=0$, we multiply $D_{46}=20622$ by $.0005+1$, and efface 5 in the extreme right hand quotient hole, knowing beforehand that the result in this case is 1.000. The product obtained is 20632.311, a number just above the dividend D_{46}. We might, if we wished, verify the result by subtracting .001 times the divisor, and we would get 20611.689,—a number just below our dividend, proving that .999 would be .001 below the true result.

* This rule is a slightly modified and more generalized form of the corresponding rule given above, and its proof is easy. Since after the black figure 9 come the red figures 8, 7, etc., in the reverse to the natural order, it is evident that $x_r = 9 + (9 - x_r) = 10 + [10 - (x_r + 2)]$, where x_r is one red figure. Now, assuming, for instance, that our red-figured number, N_r, consists of three figures, we have, $N_r = 100z_r + 10y_r + x_r = 100[10+10-(z+2)] + 10[10+10-(y+2)] + 10+10-(x+2) = 1000+1000-100z+200-200-10y-20+20-x-2 = 1000+1000-(100z+10y+x+2) = 1000+[1000-(N_r+2)]$.

Moving over the slide three stations from its extreme left position to the right, we find that five additional turns of the handle give $21663.411 > 21489 = D_{45}$. We therefore turn the handle subtractively once and obtain $21457.191 < D_{45}$, and moving over the slide one station to the left, we must turn the handle twice before obtaining the product 21498.435, just above D_{45}. We have then in our quotient holes the number 1.042, which is the required quotient corresponding to $D_{45} \div D_{46}$.

Assuming now provisionally the next difference to be again .042, we find, after turning the handle four and two additional times in the proper quotient holes, that the product obtained is $22364.559 < D_{44} = 22379$. We therefore turn the handle once more for the third decimal place, and obtain 22385.181, a number slightly above D_{44}. The corresponding quotient in the quotient holes is then found to be 1.085. Assuming again the next difference to be, like the last one, .043 or thereabout, we turn the handle the indicated number of times for each of the two corresponding quotient holes, finding the product corresponding to the difference $.044 = 23292.549 < D_{43} = 23295$. We, therefore, turn the handle once more for the third decimal place, and obtain the product 23313.171, slightly above D_{43}, giving in the quotient holes 1.0[68], the number in brackets given in red color. By the above rule for converting a red-figured number into a black-figured one, this is equivalent to 1.100+arithmetical complement of 70=1.130, which is therefore the true quotient corresponding to $D_{46} \div D_{43}$. It should be observed that we took the difference .042 and .043 as a guide in each of the subsequent cases only to shorten slightly the process of multiplication; but it was not absolutely necessary; we could have found the correct subsequent figures by turning the handle each time for the coroesponding second decimal place *tentatively* five times, and by turning the handle once subtractively on finding the corresponding product too high, as done above for obtaining the quotient $D_{45} \div D_{46}$, and as explained in the theoretical statement. Similarly, to obtain the first quotient of the series under consideration, we have availed ourselves of the fact that the dividend was in this case equal to the divisor. If this is not the case, a more literal following of the directions given in the theoretical statement for the procedure would entail but a very little extra labor in finding each of the figures of the higher orders, beginning from the left, by a tentative number of additive turns of the handle producing a product just above the corresponding dividend, followed by one subtractive turn of the handle to reduce it again to a number just below the dividend. This procedure ought now to be perfectly clear from the illustration.

It should further be noticed that there is no red zero in the "Saxonia" arithmometer, but a black zero may sometimes mean the equivalent of a red zero— namely, when the former in a quotient hole is derived from 9 by adding 9. We should, therefore, be careful to distinguish between a black zero proper and one derived in the above manner, which is equivalent to 18 also by the

above rule for conversion of red-figured numbers into black-figured ones.

Conclusion.

In conclusion I wish to say that the application of the "checking method" to serial division would also prove of considerable advantage for computing temporary and deferred annuities by means of the arithmometer. The usual continued method of arithmometer computation of these functions is based upon the formulas: $|_{n+1}a_x = |_n a_x + D_x^{-1} \times D_{x+n+1}$, and $_{n+1}| a_x = _n| a_x - D_x^{-1} \times D_{x+n+1}$, respectively,—making D_{x+n+1} the out-factor, which is quite large. If, however, we take as the basis of our computation the corresponding formulas: $|_n a_x = \frac{N_x - N_{x+n}}{D_x}$ and $_n| a_x = \frac{N_{x+n}}{D_x}$, then, beginning for any given entrance age x with the smallest dividend $N_x - N_{x+1} = D_{x+1}$ for $n=1$ and proceeding downwards to the largest dividend $N_x - N_{w-1}$, for temporary annuities, and from N_{w-1} upwards to the largest dividend N_x for deferred annuities (w being the highest age of the mortality table), we shall evidently save much labor in the continued multiplication of D_x required by our method, in which the out-factors are the differences of the required annuities themselves, since these differences begin with three significant figures and end with zero for temporary annuities, and conversely for deferred annuities, whereas the D_x's consist mostly of five significant figures. Moreover, as these differences change but gradually, we can also save labor by using provisionally for each additional multiplier a number as near the preceding multiplier as *convenient* and then correcting it so as to obtain an aggregate product just above the corresponding dividend. In practice we may also make use of such device as multiplying by the next higher unit of a given figure and subtracting the arithmetical complement of the figure times the multiplicand, provided that this arithmetical complement is smaller than the figure in the same quotient hole previously recorded, so as not to come to any *red figure* in the quotient *by subtraction.* For instance, after having found $|_1 a_{40}$ at 3% to be .962, while $|_0 a_{40} = 0$, we assume the next difference also to be approximately .962, and multiply the divisor D_{40} first by 1. and then by $-.1$, producing the aggregate multiplier (quotient) 1.862, which, however, gives a product less than N_{41}, and we must add to our multiplier .025, bringing up the aggregate multiplier to 1.887, the true value of $|_2 a_{40}$, in order to obtain an aggregateproduct just above N_{41}. We thus save the labor of multiplying the divisor originally by .9 and then changing the obtained red figure 9 to the black-figured number 1.8 by the conversion rule given above.

A still greater amount of labor can be saved if the annuities are computed only for valuation purposes where there is neither possibility nor need for great accuracy. In such a case it is much better to take as our basis for the computation the formulas: $|_n a_x = {}_1E_x + {}_2E_x + \ldots + {}_nE_x$, and $_n| a_x =$

${}_{n+1}E_x + {}_{n+2}E_x + \ldots$ etc.,—a sum of pure endowments in either case. The pure endowments, ${}_tE_x = D_{x+t} \div D_x$, are first computed to three decimal places by our "checking method" as explained above, and then these are added successively for the same value of the entrance age x, from $D_{x+1} \div D_x$ to $D_{x+n} \div D_x$ for temporary annuities, and from $D_{w-1} \div D_x$ to $D_{x+n+1} \div D_x$ for deferred annuities. Of course, in the successive addition of these endowments, themselves correct only to the third decimal place, there will of necessity arise a small accumulation of error as we proceed further from the starting point, but the maximum aggregate error cannot amount to more than a few units in the third decimal place, which for purposes of valuation is rather insignificant. The preliminary endowment tables themselves should be computed by starting for each entrance age x from the lowest dividend D_w and going upwards to the dividend D_x, by adding continually to the variable multiplier (quotient) until we reach the highest quotient 1.000. This procedure would prove a great labor-saver even in comparison with our own more accurate method of annuity computation previously explained, based upon the formulas: $|\,{}_na_x = [N_x - N_{x+n}] \div D_x$ and ${}_n|\,a_x = N_{x+n} \div D_x$, since the differences of the consecutive pure endowments are considerably smaller than the corresponding differences of the annuities themselves. This comparison in favor of the pure endowment basis of computation is especially true in the case of *decreasing* or *increasing* temporary and deferred annuities, in which the summations $\sum_{i=1}^{i=n} r_i D_{x+i}$ replace the N_x's in the numerators of the above formulas. In such a case, the preliminary computations of the Σ's, and of their differences for temporary annuities, would be wholly dispensed with, the much simpler summations of the pure endowments taking their place at the very end of the procedure.